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L^q – L^∞ Hölder continuity for quasilinear parabolic equations associated to Sobolev derivations

Fabio Cipriani^{a,*} and Gabriele Grillo^b^a *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32,
20133 Milano, Italy*^b *Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24,
10129 Torino, Italy*

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Abstract

Let X be a locally compact metrizable space endowed with a couple of equivalent finite Radon measures m and μ and let E be a Hilbert C^* -monomodule over $C_0(X)$. We consider a class of abstract nonlinear parabolic equations defined as follows. Let ∂ be a closed derivation from $L^2(X, m)$ to $L^2(E, \mu)$ and T_t be the strongly continuous nonlinear semigroup naturally associated, in the sense of Brezis (1973), to the convex l.s.c. functional $\mathcal{E}(u) = \int_X |\partial u|_x^p d\mu(x)$, where $|\cdot|$ is the natural modulus function associated to E . The generator of the semigroup considered is a natural generalization of the usual p -Laplacian operator. We suppose that a suitable Sobolev-like inequality of the form $\|u\|_{L^{2d/(d-2)}(X, m)} \leq c \|\partial u\|_{L^2(E, \mu)}$ holds true for some $d > 2$, with $p \in [2, d)$. Then T_t is a nonlinear Markov semigroup in the sense that it is order preserving and nonexpansive on each $L^q(X, m)$ for any $q \in [2, +\infty]$ and, moreover, it satisfies $\|T_t u - T_t v\|_{L^\infty(X, m)} \leq cm(X)^\alpha t^{-\beta} \|u - v\|_{L^q(X, m)}^\gamma$ for all $q \geq 2$ and suitable constants α, β, γ depending only on p, q, d . Examples include the semigroup generated by the p -Laplacian on finite measure manifolds with boundary and homogeneous Dirichlet boundary conditions, as well as by p -Laplacian-like operators associated both to regular sub-Riemannian structures, and

* Corresponding author.

E-mail addresses: fabcip@mate.polimi.it (F. Cipriani), grillo@calvino.polito.it (G. Grillo).

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to systems of (possibly singular or degenerate) vector fields satisfying the appropriate Sobolev inequalities. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction, preliminaries and statement of the main results

In a recent paper [1] we have proved an ultracontractive bound of the form

$$\|u(t)\|_{\infty} \leq c \operatorname{vol}(\Omega)^{\alpha} t^{-\beta} \|u(0)\|_q^{\gamma}, \quad t > 0, \quad u(0) \in L^q(\Omega),$$

for the solutions to a class of quasilinear parabolic equations on bounded Euclidean domains $\Omega \subset \mathbb{R}^d$, with homogeneous Dirichlet boundary conditions. The model example is the nonlinear evolution equation driven by the Euclidean p -Laplacian:

$$\dot{u} = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad (1.1)$$

with $2 \leq p < d$, the constants α , β and γ depending only on p , q , d .

One of the goals of the present work is to extend such results to several different situations, including nonlinear evolution equations governed by the p -Laplacian on manifolds, by subelliptic p -Laplacians constructed in terms of suitable vector fields on Euclidean domains or Lie groups, and by the sub-Riemannian p -Laplacian associated to a sub-Riemannian structure on a manifold.

Investigating such generalizations it became clear that the particular techniques needed to handle these different situations were special cases of a much more general and simple method applicable to a much wider environment. Therefore another achievement of the present paper is the introduction of a general analytic–algebraic point of view in analyzing the L^p regularity properties of a class of abstract nonlinear evolution equations. All our results will indeed depend on the fact that the evolution equations considered are governed by a p -Laplacian-like operator constructed from a closed derivation with values in a general Hilbert C^* -monomodule, for which an appropriate Sobolev inequality holds.

To be able to present in more detail our results, we now introduce some basic definitions. Unless otherwise stated, X will denote a locally compact metrizable space, $C_0(X)$ the algebra of (real valued) continuous functions vanishing at infinity, and m, μ a couple of equivalent finite Radon measure on X .

1.1. Hilbert C^* -modules and derivations

We now discuss a general algebraic structure whose geometric motivating example is the following. Let X be a smooth Riemannian manifold, with metric tensor g and Riemannian measure m_g , and let TX its tangent bundle. Then the

space $E = C_0(TX)$ of continuous sections of TX vanishing at infinity is clearly a module over $C_0(X)$ under pointwise multiplication in the fibers, and is endowed with a natural scalar $C_0(X)$ -valued scalar product defined by the metric g as follows: given two vector fields $\xi, \eta \in E$, define

$$\langle \xi, \eta \rangle(x) = g_x(\xi(x), \eta(x)), \quad x \in X.$$

Associated to a Radon measure μ one can consider the space of p -integrable vector fields, $L^p(TX, \mu)$.

As a generalization of the above example we shall consider a Hilbert C^* -(mono)module E over $C_0(X)$. We refer to [2] and references therein for a complete discussion. We first define the concept of inner product monomodule over $C_0(X)$. This means that there is an action of $C_0(X)$ (which can be written equivalently both on the right and on the left) over E , and that E is endowed with a bilinear symmetric map $\langle \cdot, \cdot \rangle$ from $E \times E$ to $C_0(X)$, satisfying the following properties:

- $\langle \xi, u\eta \rangle = u\langle \xi, \eta \rangle$ for all $u \in C_0(X)$, for all $\xi, \eta \in E$;
- $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in E$, and it equals zero if and only if $\xi = 0$.

We next define $|\xi| := \langle \xi, \xi \rangle^{1/2}$ for all $\xi \in E$: we shall refer to $|\xi| \in C_0(X)$ as to the *modulus* of ξ . We record here for future use that an elementary generalization of the usual Cauchy–Schwarz inequality states that

$$|\langle \xi, \eta \rangle| \leq |\xi| |\eta|$$

for all $\xi, \eta \in E$.

Finally, a Hilbert C^* -monomodule is defined as an inner product monomodule which is complete in the seminorm

$$\|\xi\|_E := \|\xi\|_\infty, \quad \xi \in E.$$

We shall use the fact that such a Hilbert C^* -module can be canonically represented as the Hilbert C^* -module of the space of continuous sections of a continuous field of Hilbert spaces (see [2, p. 152]). This latter object is essentially a continuous bundle $\{E_x: x \in X\}$ of Hilbert spaces. The running assumption imply that the dimension of the fibers E_x is an upper semicontinuous function (see [2, p. 153]), a familiar property for the local homogeneous dimension associated to families of Hörmander vector fields (see [3, p. 63]).

In complete analogy with the motivating Riemannian example we consider, for all $p \geq 1$ and all finite Radon measures μ , the spaces $L^p(E, \mu)$ defined as the completion of E under the norm

$$\|\xi\|_p := \|\xi\|_{L^p(X, \mu)}.$$

When E is represented a space of sections of Hilbert spaces, then the Hilbert space $L^2(E, \mu)$ represents likewise as a direct integral $\int_X^\oplus E_x d\mu(x)$. In particular,

if $a, b \in L^2(E, \mu)$ are represented as sections $\{a(x)\}_{x \in X}$, $\{b(x)\}_{x \in X}$, their scalar product is given by

$$(a, b)_{L^2(E, \mu)} = \int_X \langle a(x), b(x) \rangle_x d\mu(x).$$

We will also use the fact that the action of $C_0(X)$ on E extends continuously to an action on $L^p(E, \mu)$. When $p = 2$ such action can be extended to a w^* -continuous action of $L^\infty(X, \mu)$.

We remark that the above setting fits with the analysis on metric measure spaces undertaken by Cheeger in [4].

A fundamental tool in the analysis on Riemannian manifolds is the Riemannian gradient ∇ . This is a closed linear operator from $L^2(X, m_g)$ to $L^2(TX, m_g)$, whose domain is the Sobolev space $W^{1,2}(X)$. This operator is a *derivation* in the sense that it satisfies the so-called *Leibniz rule*.

To generalize such motivating setting, we will consider a closed densely defined linear operator ∂ from a dense subset $D(\partial)$ of the Hilbert space $L^2(X, m)$ to $L^2(E, \mu)$, which is required to be a derivation in the following sense.

Assumption 1.1 (Leibniz rule). The set $\mathcal{C} := D(\partial) \cap C_0(X)$ is a core for ∂ and it is an algebra such that, for all $u, v \in \mathcal{C}$, the Leibniz rule

$$\partial(uv) = u\partial v + v\partial u \quad \forall u, v \in \mathcal{C}$$

holds.

When E has the above canonical representation in terms of sections, the derivation ∂ can be written as a direct integral

$$\partial = \int_X^\oplus \partial_x d\mu(x)$$

of a family $\{\partial_x\}$ of linear maps $\partial_x : D(\partial) \rightarrow E_x$, each of which is a derivation.

1.2. The energy functional

One of the main object of the present work will be the p -energy functional $\mathcal{E}_p : L^2(X, m) \rightarrow [0, +\infty]$ defined by

$$\mathcal{E}_p(u) := \|\partial u\|_{L^p(E, \mu)}^p = \int_X |\partial u|_x^p d\mu(x)$$

on the space

$$W^{1,p}(X, \partial) := \{u \in D(\partial), \partial u \in L^p(E, \mu)\}, \quad (1.2)$$

and $+\infty$ elsewhere.

Our results will rely on the following basic assumptions.

Assumption 1.2.

- (Density) The space $W^{1,p}(X, \partial)$ is dense in $L^2(X, m)$;
- (Regularity) The space $W^{1,p}(X, \partial) \cap C_0(X)$ is dense in $W^{1,p}(X, \partial)$ in the norm

$$\|u\|_{W^{1,p}(X, \partial)} := \|u\|_{L^2(X, m)} + \mathcal{E}_p(u)^{1/p};$$

- (Sobolev inequality) There exist $d > p \geq 2$ and $C > 0$ such that

$$\|u\|_{2d/(d-2)} \leq C \mathcal{E}_2(u) \quad (1.3)$$

for all $u \in W^{1,2}(X, \partial)$.

It will be clear from the sequel that the Sobolev inequality

$$\|u\|_{rd/(d-r)} \leq C \mathcal{E}_r(u) \quad (1.4)$$

for some $d > r \geq 2$ with $p \in [r, d]$ could be alternatively assumed as well.

1.3. Lower semicontinuity

The following lower semicontinuity result for \mathcal{E}_p will allow to consider a well-defined strongly continuous nonlinear semigroup. In fact, such semigroup will be generated by the subdifferential of such functional.

Theorem 1.3. *The functional \mathcal{E}_p is convex and lower semicontinuous in the strong topology of $L^2(X, m)$, so that its subdifferential $\Delta_{p, \partial}$ is a maximally monotone operator generating a strongly continuous nonlinear contraction semigroup on $L^2(X, m)$. In addition, such semigroup define a strong solution to $\dot{u}(t) = \Delta_{p, \partial} u(t)$ for an initial data $u_0 \in L^2(X, m)$ in the sense that $u(t)$ belongs to the domain of $\Delta_{p, \partial}$ for all $t > 0$ and such equality holds for almost all $t \geq 0$.*

Proof. The convexity is clear. As concerns the lower semicontinuity of the functional at hand, we proceed as follows. Let $u_n \in W^{1,p}(X, \partial)$ is a sequence converging in $L^2(X, m)$ to a function u , consider the numerical sequence $a_n := \mathcal{E}_p(u_n)$. If $\liminf a_n = +\infty$ there is nothing to prove. Otherwise suppose that $\liminf a_n = \alpha < +\infty$ and take any subsequence u_n such that $\mathcal{E}_p(u_n) \rightarrow \alpha$. Then $\{u_n\}$ is a bounded sequence in $L^p(E, \mu)$. It is easy to show that the latter space is a normed space, this making use of the Cauchy–Schwarz inequality mentioned above. Moreover, such space is reflexive. In fact, one proves that it is uniformly convex, by showing that a Clarkson-type inequality holds true: see [5] for details. Then, possibly by passing to a subsequence, we can assume that $\{\partial u_n\}$ converges weakly in $L^p(E, \mu)$ to an element ξ of such space. Since the natural injection

of $L^p(E, \mu)$ in $L^2(E, \mu)$ is continuous, hence weakly continuous, it follows that $\{\partial u_n\}$ converges to ξ weakly in $L^2(E, \mu)$ as well. Since ∂ is a closed operator, ξ equals ∂u and, by the weak lower semicontinuity of the norm function in any Banach space, we have $\mathcal{E}_p(u) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}_p(u_n)$; hence the assertion.

The other statements follow from general arguments: see [6,7]. \square

1.4. Nonlinear Markov semigroups

Some of the general theory developed in [5] (see also [8,9]) will be crucial to the present work. In fact, in such paper a notion of *nonlinear Dirichlet form* has been given as follows. Let $\mathcal{E} : L^2(X, m) \rightarrow [0, +\infty]$ be a lower semicontinuous, convex functional finite on a dense subset of $L^2(X, m)$. Define the following closed and convex sets:

$$C_1 := \{(u, v) \in L^2(X, m) \oplus L^2(X, m) : u \leq v\},$$

$$C_2(\alpha) := \{(u, v) \in L^2(X, m) \oplus L^2(X, m) : \|u - v\|_\infty \leq \alpha\}$$

for any positive α .

It is shown in [5] that the projection P_1 onto C_1 is given by

$$P_1(u, v) = \begin{cases} (u, v) & \text{if } u \leq v, \\ (\frac{1}{2}(u+v), \frac{1}{2}(u+v)) & \text{if } u > v \end{cases}$$

for all $(u, v) \in L^2(X, m) \oplus L^2(X, m)$, and that the projection $P_{2,\alpha}$ onto $C_2(\alpha)$ is given by

$$P_{2,\alpha}(u, v) = \begin{cases} (u, v) & \text{if } |u - v| \leq \alpha, \\ (\frac{1}{2}(u+v-\alpha), \frac{1}{2}(u+v+\alpha)) & \text{if } u - v < -\alpha, \\ (\frac{1}{2}(u+v+\alpha), \frac{1}{2}(u+v-\alpha)) & \text{if } u - v > \alpha. \end{cases}$$

Theorem 1.4 [5]. *Let $\{T_t := \exp[tA] : t \geq 0\}$ be the strongly continuous one-parameter (nonlinear) semigroup, associated to the the subdifferential A of a convex lower semicontinuous functional \mathcal{E} . Then it satisfies the condition*

$$(\mathcal{E} \oplus \mathcal{E})(P_1(w)) \leq (\mathcal{E} \oplus \mathcal{E})(w)$$

for any $w \in L^2(X, m) \oplus L^2(X, m)$ if and only if it is order preserving sense that $T_t u \leq T_t v$ for all $t \geq 0$. It satisfies the condition

$$(\mathcal{E} \oplus \mathcal{E})(P_{2,\alpha}(w)) \leq (\mathcal{E} \oplus \mathcal{E})(w),$$

for any $w \in L^2(X, m) \oplus L^2(X, m)$ if and only if it is nonexpansive in $L^q(X, m)$ for all $q \in [2, +\infty]$; i.e., $T_t(L^q(X, m)) \subset L^q(X, m)$ for all $t > 0$ and $q \in [2, +\infty]$, and $\|T_t u - T_t v\|_q \leq \|u - v\|_q$ for all such q and all $u, v \in L^q(X, m)$.

A semigroup satisfying the conditions of the above theorem is said to be a *nonlinear Markov semigroup*. The corresponding functional is said to be a *nonlinear Dirichlet form*.

1.5. The main result

We are now ready to state our main result on L^q – L^∞ Hölder continuity for the solutions to the evolution equation at hand.

Theorem 1.5. *Under the running assumptions, let us consider the evolution equation $\dot{u} = \Delta_{p,\partial} u$ associated to the subdifferential $\Delta_{p,\partial}$ of the convex lower semicontinuous functional \mathcal{E}_p , with $2 \leq p < d$, corresponding to initial data in $L^{q_0}(X, m)$ for some $q_0 \in [2, +\infty]$.*

Then the evolution considered is Markovian, in the sense that if $u(\cdot)$ and $v(\cdot)$ are solutions corresponding to the initial data u_0 and v_0 in $L^2(X, m)$, then $u_0 \leq v_0$ implies $u(t) \leq v(t)$ for all t and, moreover, $\|u(t) - v(t)\|_{L^q(X, m)} \leq \|u_0 - v_0\|_{L^q(X, m)}$ for all $q \in [2, +\infty]$.

Moreover, the following $L^{q_0}(X, m)$ – $L^\infty(X, m)$ Hölder continuity holds:

$$\|u(t) - v(t)\|_{L^\infty(X, m)} \leq C \frac{m(X)^\alpha}{t^\beta} \|u_0 - v_0\|_{L^{q_0}(X, m)}^\gamma. \quad (1.5)$$

The constants α , β , γ are defined as

$$\begin{aligned} \alpha &= \frac{d-p}{d} \left[1 - \left(\frac{q_0}{q_0 + p - 2} \right)^{d/p} \right], \\ \beta &= \frac{1}{p-2} \left[1 - \left(\frac{q_0}{q_0 + p - 2} \right)^{d/p} \right], \\ \gamma &= \left(\frac{q_0}{q_0 + p - 2} \right)^{d/p}. \end{aligned} \quad (1.6)$$

In particular, the evolution considered is L^{q_0} – L^∞ smoothing, in the sense that

$$\|u(t)\|_{L^\infty(X, m)} \leq C \frac{|N|^\alpha}{t^\beta} \|u_0\|_{L^{q_0}(X, m)}^\gamma. \quad (1.7)$$

It should be noticed that, as $p \rightarrow 2$, $\alpha(p) \rightarrow 0$, $\beta(p) \rightarrow d/(2q_0)$ and $\gamma(p) \rightarrow 1$, as expected from the well-known ultracontractive estimate for the solutions to linear evolution equations driven by uniformly elliptic second-order differential operators in divergence form (cf. [10] and references therein):

$$\|u(t)\|_\infty \leq \frac{C}{t^{\dim M/(2q_0)}} \|u_0\|_{q_0}.$$

Results of the form (1.1) have been recently obtained in [1] for parabolic evolutions governed by operators of p -Laplacian type on Euclidean domains and with coefficients satisfying Carathéodory-type assumptions. It should be noted that even in such situation the present results are more general, since no analogue of (1.5) was proved in [1], and the results given there for the Euclidean p -Laplacian hold for L^q initial data with $q \geq q_0(p) > 2$ sufficiently large.

It is a well-known fact that the assumed Sobolev inequalities are a consequence of diagonal estimates on the heat kernel. Thus, we record in the following corollary the surprising fact that *linear ultracontractivity* for the heat semigroup implies the validity of what we could call *nonlinear ultracontractivity*, in the sense of Theorem 1.5, for the nonlinear heat equations governed by the associated p -Laplacians for $p \in (2, d)$.

Corollary 1.6. *Let $-\Delta_{2,\beta}$ be the (linear) positive self-adjoint operator associated with the (linear) Dirichlet form \mathcal{E}_2 . Assume that Assumption 1.1 holds, and that the density and the regularity condition of Assumption 1.2 hold as well. Let S_t be the corresponding semigroup and assume that the ultracontractive estimate*

$$\|S_t u\|_\infty \leq C t^{-d/4} \|u\|_2 \quad \forall t > 0, \forall f \in L^2(M)$$

holds for some $d > 2$. Let also $T_t^{(p)}$ be the nonlinear semigroup associated to the functional \mathcal{E}_p . Then all the assertions of Theorem 1.5 hold for the semigroup $T_t^{(p)}$ for any $p \in [2, d)$, so that such semigroups are Markov semigroups and they satisfy (1.5).

Proof. It suffices to notice that the running assumption imply, by [3, Theorem II.5.2] (or by the work of Davies [10]), the validity of the Sobolev inequality $\|f\|_{2d/(d-2)}^2 \leq C \mathcal{E}_2(f)$. \square

We finally comment that Sobolev inequalities are also well known to be equivalent, for example, in the context of connected unimodular Lie groups, to lower bounds for the volume $|B(x, r)|$ of intrinsic balls $B(x, r)$ of the form $|B(x, t)| \geq C t^d$: the constant d in that estimate is the same appearing in the resulting Sobolev inequality (see [3, p. 3]). This gives an alternative characterization of our assumptions, whose details can be found in [3].

It is a pleasure to thank the referee for his (or her) careful reading of the first version of the present manuscript, and for pointing our attention on several useful references.

2. Examples

The basic examples we have in mind are the following:

2.1. The p -Laplacian on manifolds

Let M be a smooth connected, orientable manifold without boundary with Riemannian metric g and Riemannian volume form m . Let N be a open submanifold

with smooth boundary and finite measure. Consider, for $p \geq 2$, the functional \mathcal{E}_p given by

$$\mathcal{E}_p(u) := \int_N |\nabla u|^p \, dm$$

on the usual Sobolev space $W_0^{1,p}(N)$, where ∇ is the Riemannian gradient, $|\cdot|$ is the Riemannian length. The associated generator is proportional, on smooth compactly supported functions, to the Riemannian p -Laplacian operator

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

This functional falls within our discussion because ∇ is a closed derivation from $L^2(N, m)$ to the monomodule, over $C_0(N)$, of bounded continuous sections of the tangent bundle TN , with domain the Sobolev space $W_0^{1,2}(N)$. The Riemannian metric tensor defines the required $C_0(N)$ -valued scalar product.

The appropriate Sobolev inequalities are well known if $p \in [2, n]$, where n is the dimension of M . In particular, the parameter d appearing in our discussion coincides with n . For general reference on Sobolev inequalities on manifolds see [11, 12] and references therein.

Corollary 2.1. *Let $u(t)$, $v(t)$ be solutions to the equation*

$$\dot{u} = \Delta_p u$$

corresponding to the initial data u_0, v_0 , with homogeneous Dirichlet boundary conditions on ∂N . Then Theorem 1.5 and, in particular, inequalities (1.5) and (1.7) hold, with the choice $d = n$, whenever $p \in [2, n]$.

2.2. Regular sub-Riemannian structures

Let M be again a smooth, connected, orientable manifold without boundary. Consider a *distribution* on M , that is a smooth subbundle of TM , say \mathcal{D} , such that the Lie algebra generated by \mathcal{D} at any point $m \in M$ coincides with the tangent space $T_m M$. A Riemannian metric on \mathcal{D} is a C^∞ real function on \mathcal{D} such that each restriction of g on the fibers $\mathcal{D}(m)$ is a positive definite quadratic form. A *sub-Riemannian structure* on M is a couple (\mathcal{D}, g) , where \mathcal{D} is a distribution on M and g a Riemannian metric on \mathcal{D} (see [13–15] and references therein).

For any point $m \in M$ we define, by induction on k in $\mathbb{N} \setminus \{0\}$, the nondecreasing sequence $\{\mathcal{D}^k\}_{k \in \mathbb{N} \setminus \{0\}}$ of subspaces of $T_m M$ as follows: $\mathcal{D}^1(m) := \mathcal{D}(m)$ and, for $k \geq 2$,

$$\mathcal{D}^k(m) := [\mathcal{D}(m), \mathcal{D}^{k-1}(m)] := \{[V, W] : V \in \mathcal{D}(m), W \in \mathcal{D}^{k-1}(m)\}.$$

A point $m \in M$ is said to be *regular* if the functions $m \in M \mapsto \dim \mathcal{D}^n(m)$ are constant in a neighbourhood of m for all $n \in \mathbb{N} \setminus \{0\}$. A sub-Riemannian structure is said to be regular if every point is regular.

Finally, given a smooth function f on M , let $d_{\mathcal{D}}f : \mathcal{D} \rightarrow \mathbb{R}$ be the restriction to \mathcal{D} of the usual differential $df : TM \rightarrow \mathbb{R}$. By defining g^* to be the metric naturally associated to g on the cotangent space T^*M , we can thus define the quantity $\langle d_{\mathcal{D}}f(m) \rangle_{g^*}$ to be the length, in the metric g^* , of $d_{\mathcal{D}}f(m)$.

Let us choose a volume form ν on M , and consider in the sequel all Lebesgue and Sobolev spaces w.r.t. to such form. In particular, the Sobolev space $W_{0,\mathcal{D}}^{1,p}(M)$ will be the closure of the set of smooth, compactly supported functions on M , under the norm

$$\|u\|_{1,p}^p := \|u\|_p^p + \|\langle d_{\mathcal{D}}f \rangle_{g^*}\|_p^p.$$

Sobolev spaces on any region $N \subset M$ are defined likewise.

Then we shall consider any solution (which is not required to be positive) to the evolution equation formally written as

$$\dot{u} = -d_{\mathcal{D}}^*(\langle d_{\mathcal{D}}f \rangle_{g^*}^{p-2} d_{\mathcal{D}}u) := \Delta_{p,\mathcal{D}}u \quad (2.1)$$

with homogeneous Dirichlet boundary conditions on ∂N , where $N \subset M$ is a domain with smooth boundary and finite measure. The operator $\Delta_{p,\mathcal{D}}$ will be called *sub-Riemannian p -Laplacian*.

To give sense to the such operator and to the quasilinear parabolic differential equation (1.1) we shall consider, for any $p \geq 2$, the functional given by

$$\mathcal{E}_{p,\mathcal{D}}(u) = \int_N \langle d_{\mathcal{D}}u \rangle_{g^*}^p d\nu \quad (2.2)$$

on the Sobolev space $W_{0,\mathcal{D}}^{1,p}(N)$ and whose value is $+\infty$ otherwise in $L^2(N)$, so that it is finite on a dense set in $L^2(N)$. By analogy with the linear case $p = 2$, we take the attitude to interpret the domain of definition by saying that *homogeneous* Dirichlet boundary conditions are imposed at ∂N .

The monomodule E over $C_0(N)$ is the space of bounded continuous sections of \mathcal{D} . The metric g^* defines the $C_0(N)$ -valued inner product on continuous sections of \mathcal{D} . The required properties for the derivation $d_{\mathcal{D}}$ are well known.

By well-known results [13,14], in the regular case the following isoperimetric inequalities hold:

$$m_{d_H-1}(\partial\Omega) \geq C(m_{d_H}(\Omega))^{(d_H-1)/d_H}$$

for any domain $\Omega \subset M$ with smooth boundary, where m_{d_H} (respectively, m_{d_H-1}) denotes the Hausdorff measure of order d_H (respectively, $d_H - 1$) and d_H the Hausdorff dimension of M when endowed with the metric corresponding to the sub-Riemannian structure. In fact, we refer to [16] for isoperimetric inequalities on the Heisenberg group and to [17,18] for isoperimetric inequalities in the setting of singular sub-Riemannian structures associated to locally Lipschitz vector fields.

From such inequalities the validity of the Sobolev inequalities

$$\|f\|_{pd_H/(d_H-p)} \leq C_p \|\langle d_{\mathcal{D}} f \rangle_{g^*}\|_p,$$

for all smooth compactly supported functions f on M and $p \in [1, d_H)$, follow.

Corollary 2.2. *Let $u(t), v(t)$ be solutions to Eq. (2.1), corresponding to the initial data u_0, v_0 , with homogeneous Dirichlet boundary conditions on ∂N . Then Theorem 1.5 and, in particular, inequalities (1.5) and (1.7) hold with the choice $d = d_H$, whenever $p \in [2, d_H)$.*

2.3. Hörmander systems on nilpotent Lie groups

Let G be a nilpotent Lie group and $\{X_i\}_{i=1}^s$ be a Hörmander system of left-invariant vector fields on G which are linearly independent at the origin. The sub-Riemannian structure on G is defined by requiring $\mathcal{D}(m)$ to be the linear span of the vectors $X_1(m), \dots, X_s(m)$ and g_m to be the metric on $\mathcal{D}(m)$ such that $g_m(a_1 X_1(m) + \dots + a_s X_s(m)) := a_1^2 + \dots + a_s^2$, so that for any smooth function f on G

$$\langle d_{\mathcal{D}} f \rangle_{g^*}^2 = |X_1 f|^2 + \dots + |X_s f|^2.$$

Define d_1 to be the *local dimension* of G , so that $V(t) \sim t^{d_1}$ for small t , $V(t)$ being the Haar measure of the intrinsic ball of radius t relative to the system $\{X_i\}_{i=1}^m$ and centered at the identity. Similarly, let d_2 be the dimension at infinity of G ; we start supposing that $2 \leq d_1 \leq d_2$. Then the Sobolev inequality

$$\|f\|_{2d/(d-2)} \leq C \|Xf\|_2$$

holds for any $d \in [d_1, d_2]$, provided in addition $d > 2$. Here we have defined $Xf := (X_1 f, \dots, X_s f)$. In any case, the Sobolev inequality (1.3) holds when d is any constant not smaller than d_1 . For this results see p. 56 of [3].

We comment that the subelliptic p -Laplacian naturally associated to a collection of vector fields has appeared, in connection with different problems, in several papers among which we quote without claim of completeness the basic papers [17] and [19] and, as later references, [19–21] and references therein. We also comment that heat kernel bounds for the *linear* sub-Laplacian associated to a collection of vector fields are given in [22].

Let now N be a smooth bounded domain in G and let $W_{0,X}^{1,p}(N)$ be the Sobolev space defined according to the corresponding sub-Riemannian structure as in the previous example. Define also the functional

$$\mathcal{E}_{p,X}(u) = \int_N |Xu|^p \, dg$$

for $u \in W_{0,X}^{1,p}(N)$, and $+\infty$ elsewhere in $L^2(N, dg)$. Its subdifferential will be denoted by $\Delta_{p,X}$.

Corollary 2.3. *Let $u(t), v(t)$ be solutions to the equation*

$$\dot{u} = \Delta_{p,X} u$$

corresponding to the initial data u_0, v_0 , with homogeneous Dirichlet boundary conditions on ∂N . Then Theorem 1.5 and, in particular, inequalities (1.5) and (1.7) hold for any choice of d in the range $[d_1, d_2]$ and for any $p \in [2, d)$, where d_1 and d_2 are, respectively, the local dimension and the dimension at infinity of G , whenever the inequality $2 \leq d_1 \leq d_2$ is assumed in addition.

Similar corollaries can be stated with the obvious changes in any of the forthcoming examples.

2.4. Hörmander systems on unimodular groups

We generalize now the above setting by letting G be a connected unimodular Lie group and $\{X_i\}_{i=1}^m$ a Hörmander system on G as in the previous example. The corresponding sub-Riemannian structure is constructed as above. Let d_1 be the local dimension of G . Then the Sobolev inequality (1.3) holds with $d = d_1$ provided $d > 2$. For this result see p. 70 of [3].

2.5. Hörmander systems on nonunimodular groups

Let G be a connected nonunimodular Lie group and let $\{X_i\}_{i=1}^m$ and the corresponding sub-Riemannian structure be as above. We consider a right invariant Haar measure dx and the left invariant Haar measure $dv = m dx$, where m is the modular function. Then the Sobolev inequality (1.3) holds (with the norms taken w.r.t. ν) choosing d to be any constant not smaller than d_1 , the local dimension of G , provided one uses the above norms: see p. 127 of [3].

2.6. Systems of vector fields with locally Lipschitz coefficients

Let $\{X_i\}_{i=1}^n$ be a systems of vector fields with locally Lipschitz coefficients on an open and bounded set $\Omega \subset \mathbb{R}^d$. Suitable choices of such families give rise to examples of *singular sub-Riemannian structures*, because the regularity condition for the induced sub-Riemannian structure may fail. We shall make no attempt to give a complete description, or even a complete bibliographic reference, of the existing Sobolev inequalities associated to the family $\{X_i\}$ under the various possible conditions to be imposed on the vector fields at hand, on the domain considered and on the class of functions for which the Sobolev inequality should hold. We only mention, without no claim of completeness, the works of Capogna, Danielli, Franchi, Gallot, Garofalo, Lu, Maheaux, Saloff-Coste, Wheeden, and, in particular, the papers [17, 18, 21, 23–29] and reference therein. We also refer to [30] for a more complete list of references.

We shall discuss now in somewhat more detail some results of [18] as concerns weighted Sobolev inequalities for the system of vector fields considered. Assume that u and v are nonnegative functions locally integrable in Ω with respect to Lebesgue measure. Assume that $u(x)dx$ is a doubling measure and that v belongs to the Muckenhoupt class A_p for some $p \geq 1$. Fix a suitable ball $B_0 := B(x_0, t_0) \subset \Omega$, where the balls are meant with respect to the well-known intrinsic pseudometric associated to the vector fields at hand, which will be assumed to be a true metric without further comment. Assume also that, for some $q > p \geq 1$, the balance condition

$$\frac{r}{r_0} \left(\frac{u(B_r)}{u(B_0)} \right)^{1/q} \leq C \left(\frac{v(B_r)}{v(B_0)} \right)^{1/p}$$

holds for any ball $B_r = B(X, r) \subset B(x_0, 5r_0)$, where quantities like $u(B_r)$ mean the integral of u over B_r . Then, under suitable assumptions on the system of vector fields, the weighted Sobolev inequality

$$\begin{aligned} & \left(\frac{1}{u(B_0)} \int_{B_0} |f(x)|^q u(x) dx \right)^{1/q} \\ & \leq C r_0 \left(\frac{1}{v(B_0)} \int_{B_0} \left(\sum_{i=1}^n |X_i f(x)|^2 \right)^{p/2} v(x) dx \right)^{1/p} \end{aligned} \quad (2.3)$$

for all Lipschitz continuous functions g compactly supported in B_0 . The class of vector fields for which the stated conclusion holds includes at least the case of Hörmander vector fields, the case in which $X_i = \lambda_i \partial_i$ for all i , where λ_i are suitable Lipschitz functions satisfying certain integral conditions, which essentially amount to requiring that one can reach a sufficiently large part of a metric ball starting from its center by means of suitable sub-unit curves, and the Grushin-type operators considered in [25].

A closer inspection of the proof of (2.3) shows that it is based only on a suitable representation formula proved in [31], whose proof depends in turn only on a L^1 – L^1 Poincaré-type inequality for the vector fields at hand and on an assumption on the volume of intrinsic balls, which was later relaxed to an even more natural one in [32].

3. Proof of the theorem

The proof will be divided into several steps. First, the Markov property of the nonlinear semigroup considered will be proved starting from the main results of [5]. The Sobolev inequalities assumed throughout will be used to derive a family of logarithmic Sobolev inequalities. Then a differential inequality for the quantity $\|u(t)\|_{r(t)}$ will be obtained, by making use of recent results of [5] concerning

the Markovianity of the nonlinear evolution considered and of suitable numerical inequalities (see [33]). The use of the above mentioned logarithmic Sobolev inequality and of convexity arguments will allow to integrate the differential inequality at hand and to arrive, via limiting arguments and, again, the Markovianity of the evolution, to the stated bounds.

Step 1. The Markov property

Since convexity and lower semicontinuity of the functional \mathcal{E}_p has already been proved, we only have to verify the required contraction properties.

As concerns the contraction property of $\mathcal{E}_p \oplus \mathcal{E}_p$ w.r.t. P_1 we first show that $\mathcal{E}_p \oplus \mathcal{E}_p$ is finite on $P_1(u, v)$ for all u, v belonging to the domain $W^{1,p}(X, \partial)$. In fact, we shall use the following *chain rule*: if $w \in W^{1,p}(X, \partial) \cap C_0(X)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function vanishing at the origin and with bounded derivative, then

$$\partial f(w) = f'(w) \partial w.$$

This follows from Lemma 7.2 in [34], where a more general version for derivations on bimodules is given. When applied to the special case of derivations on monomodules, it gives the above equality. Then

$$\mathcal{E}_p(f(w)) \leq \|f'\|_{C_0(\mathbb{R})} \mathcal{E}_p(w).$$

By the regularity assumption and the lower semicontinuity of \mathcal{E}_p the same property and estimate hold for $w \in W^{1,p}(X, \partial)$. Let now $g(x) = x \vee 0$, and choose f_n to be a sequence of C^1 functions vanishing at the origin, with $|f'| \leq 1$ and such that $f_n(w) \rightarrow g(w)$ in $L^2(X, m)$ given $w \in W^{1,p}(X, \delta)$ (see [35, p. 8]). Then, since

$$\mathcal{E}_p(f_n(w)) \leq \mathcal{E}_p(w),$$

the lower semicontinuity of \mathcal{E}_p implies that

$$\mathcal{E}_p(g(w)) \leq \mathcal{E}_p(w) < +\infty.$$

Now notice that we can write

$$P_1(u, v) = \left(u - \frac{u \wedge v}{2}, v + \frac{u \vee v}{2} \right) = \left(u - \frac{1}{2}g(u-v), v + \frac{1}{2}g(u-v) \right)$$

for all $u, v \in L^2(X, m)$; it then follows that $\mathcal{E}_p(P_1(u, v))$ is finite.

We can assume that the functions f_n above also satisfy, besides the preceeding assumptions, the following ones: $f_n(s) = s$ if $s > 0$, $f_n(s) = -1/n$ if $s \leq -2/n$, $|f_n(s)| \leq -1/n$ if $-2/n \leq s < 0$. Define

$$P_1^n(u, v) = \left(u - \frac{f_n(u-v)}{2}, v + \frac{f_n(u-v)}{2} \right)$$

and the regions

$$A := \{u \geq v\}, \quad B_n := \left\{ -\frac{2}{n} < u - v < 0 \right\}, \quad C_n := \left\{ u - v \leq -\frac{2}{n} \right\}.$$

Then $X = A \cup B_n \cup C_n$ for all n and, by using the convexity of the functional at hand in the penultimate step:

$$\begin{aligned} \mathcal{E}_p \oplus \mathcal{E}_p(P_1^n(u, v)) &= \mathcal{E}_p\left(u - \frac{f_n(u-v)}{2}\right) + \mathcal{E}_p\left(v + \frac{f_n(u-v)}{2}\right) \\ &= \int_X \left| \partial u - \frac{1}{2} f'_n(u-v) \partial(u-v) \right|^p d\mu \\ &\quad + \int_X \left| \partial v + \frac{1}{2} f'_n(u-v) \partial(u-v) \right|^p d\mu \\ &= \int_A \left| \partial \left(\frac{u+v}{2} \right) \right|^p d\mu + \int_{C_n} |\partial u|^p d\mu \\ &\quad + \int_{B_n} \left| \partial u - \frac{1}{2} f'_n(u-v) \partial(u-v) \right|^p d\mu + \int_A \left| \partial \left(\frac{u+v}{2} \right) \right|^p d\mu \\ &\quad + \int_{C_n} |\partial v|^p d\mu + \int_{B_n} \left| \partial v + \frac{1}{2} f'_n(u-v) \partial(u-v) \right|^p d\mu \\ &\leq \int_{A \cup C_n} (|\partial u|^p + |\partial v|^p) d\mu + K \int_{B_n} (|\partial u|^p + |\partial v|^p) d\mu \\ &\rightarrow \mathcal{E}_p \oplus \mathcal{E}_p(u, v) \end{aligned}$$

as $n \rightarrow +\infty$, because the measure of B_n tends to zero as n tends to infinity. The contraction property then follows again by the lower semicontinuity of \mathcal{E}_p because $P_1^n(u, v)$ converges to $P(u, v)$ in $L^2(X, m) \oplus L^2(X, m)$.

The contraction property relative to $P_{2,\alpha}$ is proved likewise, by only using the fact that in addition that $d_P 1 = 0$.

Step 2. Logarithmic Sobolev inequalities from Sobolev inequalities

It is a standard fact (see, e.g., [3, p. 2]; the arguments used there for the Euclidean situation are applicable to the present setting as well) that the validity of Sobolev inequalities in $W^{1,2}(X, \partial)$ also implies the validity of the inequality

$$\|f\|_{L^{pd/(d-p)}(X, m)} \leq C \mathcal{E}_p(f) \quad (3.1)$$

for all $f \in C_0(X) \cap W^{1,p}(X, \partial)$ and all $p \in (2, d)$.

We now adapt a well-known argument to prove that an ordinary Sobolev inequality implies a family of Gross's logarithmic Sobolev inequalities [36]. Let ν be the probability measure $\nu = f^p dm$, where f is a positive function of unit $L^p(X, m)$ norm. Then, taking below $q = p^2/(d - p)$ and C being the constant appearing in (3.1):

$$\begin{aligned} \int_X f^p \log f \, dm &= \int_X \log f \, d\nu \leq \frac{1}{q} \log \int_X f^q \, d\nu = \frac{d}{p^2} \log \|f\|_{L^{dp/(d-p)}(X, m)}^p \\ &\leq \frac{d}{p^2} (-\log \varepsilon + \varepsilon \|f\|^p L^{dp/(d-p)}(X, m)) \\ &\leq \frac{d}{p^2} (-\log \varepsilon + \varepsilon C \mathcal{E}_p(f)), \end{aligned}$$

where the first inequality follows from Jensen's inequality (since \log is concave), the second one from the numerical inequality $\log t < t$, the last one from the Sobolev inequality, and $\varepsilon > 0$ is arbitrary. By homogeneity one obtains the following family of logarithmic Sobolev inequalities, valid for any $\varepsilon > 0$:

$$\begin{aligned} \int_X |f|^p \log |f|^p \, d\nu - \left(\int_N |f|^p \, d\nu \right) \log \left(\int_N |f|^p \, d\nu \right) \\ \leq \frac{d}{p} [\|f\|_p^p (-\log \varepsilon) + \varepsilon C \mathcal{E}_p(f)], \end{aligned} \quad (3.2)$$

where $2 \leq p < d$ and C is the constant appearing in the Sobolev inequality (3.1). This is achieved first for positive functions in $W^{1,p}(X, \partial)$, but since $\mathcal{E}_p(|u|) \leq \mathcal{E}_p(u)$, the same inequalities also hold without the positivity assumption.

Step 3. Time differentiability of Lebesgue norms of the solution

We shall use the Markov property proved in Step 1. In fact, take initial data $u_0, v_0 \in L^\infty(N)$ and for any $r \geq 2$ consider the function $f_r : (0, \infty) \rightarrow [0, \infty)$:

$$f_r(s) = \|u(s) - v(s)\|_r^r,$$

where $u(s), v(s)$ are the solutions corresponding to the initial data u_0, v_0 and the norms are taken with respect to the measure m . We now prove that f_r is a.e. differentiable and

$$\dot{f}_r(s) = -r(r-1) \int_X |u(s) - v(s)|^{r-2} |\partial(u(s) - v(s))|^p \, d\mu. \quad (3.3)$$

The fact that f is well-defined follows from the above mentioned Markovianity of the evolution equation and from the fact that X has finite m -measure. To prove (3.3) we notice that, by the Leibniz rule, we have that for a.e. t , letting $\Delta_{p,\partial}$ be the subdifferential of \mathcal{E}_p ,

$$\begin{aligned}
\frac{d}{ds} f_r(s) &= r \int_X |u(s) - v(s)|^{r-1} \operatorname{sgn}(u(s) - v(s)) (\dot{u}(s) - \dot{v}(s)) \, d\mu \\
&= r \int_X |u(s) - v(s)|^{r-1} \operatorname{sgn}(u(s) - v(s)) \\
&\quad \times [\Delta_{p,\partial}(u(s)) - \Delta_{p,\partial}(v(s))] \, d\mu \\
&= -r \int_X \langle \partial(|u(s) - v(s)|^{r-1}) \operatorname{sgn}(u(s) - v(s)), \\
&\quad |\partial u(s)|^{p-2} \partial u(s) - |\partial v(s)|^{p-2} \partial v(s) \rangle \, d\mu \\
&= -r(r-1) \int_X (|u(s) - v(s)|^{r-2} \partial(u(s) - v(s)), \\
&\quad |\partial u(s)|^{p-2} \partial u(s) - |\partial v(s)|^{p-2} \partial v(s)) \, d\mu.
\end{aligned}$$

To proceed further, we shall use a well-known inequality (see [33]), valid in any Hilbert space: for all $x \in X$, if $a, b \in E_x$ and $p \geq 2$, then

$$\langle |a|_x^{p-2} a - |b|_x^{p-2} b, a - b \rangle_x \geq c |a - b|_x^p \quad (3.4)$$

for some positive constant c . We can then apply such inequalities to show that, c denoting such constant,

$$\begin{aligned}
\frac{d}{ds} f_r(s) &= -r(r-1) \int_X (|u(s) - v(s)|^{r-2} \partial(u(s) - v(s)), \\
&\quad |\partial u(s)|^{p-2} \partial u(s) - |\partial v(s)|^{p-2} \partial v(s)) \, d\mu \\
&\leq -cr(r-1) \int_X |u(s) - v(s)|^{r-2} |\partial(u(s) - v(s))|^p \, d\mu.
\end{aligned}$$

Let now $r : [0, +\infty) \rightarrow [2, +\infty)$ be a monotonically nondecreasing C^1 function. We now compute the time derivative of the function $s \mapsto \|u(s) - v(s)\|_{r(s)}^{r(s)}$ and prove that, for almost all s ,

$$\begin{aligned}
\frac{d}{ds} \|u(s) - v(s)\|_{r(s)}^{r(s)} &= \dot{r}(s) \int_X |u(s) - v(s)|^{r(s)} \log |u(s) - v(s)| \, d\mu \\
&\quad - r(s)(r(s) - 1) \int_X |u(s) - v(s)|^{r(s)-2} |\partial(u(s) - v(s))|^p \, d\mu. \quad (3.5)
\end{aligned}$$

In fact, we have

$$\begin{aligned}
& \frac{d}{ds} \|u(s) - v(s)\|_{r(s)}^{r(s)} \\
&= \dot{r}(s) \frac{\partial}{\partial r} \|u(s) - v(s)\|_r^r \Big|_{r=r(s)} + \frac{\partial}{\partial s} \|u(s) - v(s)\|_r^r \Big|_{r=r(s)} \\
&\leq \dot{r}(s) \int_X |u(s) - v(s)|^{r(s)} \log |u(s) - v(s)| \, d\mu \\
&\quad - cr(s)(r(s) - 1) \int_X |u(s) - v(s)|^{r(s)-2} |\partial(u(s) - v(s))|^p \, d\mu,
\end{aligned}$$

where we have used the previous calculations to write the derivative w.r.t. s and the fact that

$$\begin{aligned}
\frac{\partial}{\partial r} \|u(s) - v(s)\|_r^r &= \frac{\partial}{\partial r} \int_X |u(s) - v(s)|^r \, d\mu = \int_X \frac{\partial}{\partial r} e^{r \log |u(s) - v(s)|} \, d\mu \\
&= \int_X |u(s) - v(s)|^r \log |u(s) - v(s)|^r \, d\mu.
\end{aligned}$$

The above result will now be used to compute the time derivative of the function $s \mapsto \|u(s) - v(s)\|_{r(s)}$ and prove that, for almost all s ,

$$\begin{aligned}
& \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\
&\leq \frac{\dot{r}(s)}{r(s)} \int_X \frac{|u(s) - v(s)|^{r(s)}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \log \frac{|u(s) - v(s)|}{\|u(s) - v(s)\|_{r(s)}} \, d\mu \\
&\quad - c \frac{(r(s) - 1)}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \int_X |u(s) - v(s)|^{r(s)-2} |\partial(u(s) - v(s))|^p \, d\mu.
\end{aligned} \tag{3.6}$$

In fact,

$$\begin{aligned}
& \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\
&= \frac{d}{ds} r(s)^{-1} \log \|u(s) - v(s)\|_{r(s)}^{r(s)} \\
&= -\frac{\dot{r}(s)}{r(s)^2} \log \|u(s) - v(s)\|_{r(s)}^{r(s)} + r(s)^{-1} \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)}^{r(s)} \\
&= -\frac{\dot{r}(s)}{r(s)} \log \|u(s) - v(s)\|_{r(s)} + \frac{r(s)^{-1}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \frac{d}{ds} \|u(s) - v(s)\|_{r(s)}^{r(s)} \\
&\leq -\frac{\dot{r}(s)}{r(s)} \log \|u(s) - v(s)\|_{r(s)} + \frac{r(s)^{-1}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\dot{r}(s) \int_X |u(s) - v(s)|^{r(s)} \log |u(s) - v(s)| \, d\mu \right. \\
& \quad \left. - cr(s)(r(s) - 1) \int_X |u(s) - v(s)|^{r(s)-2} |\partial(u(s) - v(s))|^p \, d\mu \right) \\
& = \frac{\dot{r}(s)}{r(s)} \int_X \frac{|u(s) - v(s)|^{r(s)}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \log \frac{|u(s) - v(s)|}{\|u(s) - v(s)\|_{r(s)}} \, d\mu \\
& \quad - c \frac{(r(s) - 1)}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \int_X |u(s) - v(s)|^{r(s)-2} |\partial(u(s) - v(s))|^p \, d\mu.
\end{aligned}$$

We can also restate the above result in terms of the functional \mathcal{E}_p . In fact, the following inequality holds true for almost all s :

$$\begin{aligned}
& \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\
& \leq \frac{\dot{r}(s)}{r(s)} \int_X \frac{|u(s) - v(s)|^{r(s)}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \log \frac{|u(s) - v(s)|}{\|u(s) - v(s)\|_{r(s)}} \, d\mu \\
& \quad - c \left(\frac{p}{r + p - 2} \right)^p \frac{(r(s) - 1)}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \mathcal{E}_p \left(|u(s) - v(s)|^{(r(s)+p-2)/p} \right).
\end{aligned} \tag{3.7}$$

We finally combine the latter inequality with the logarithmic Sobolev inequalities of Step 3 to obtain an inequality in which the functional \mathcal{E} does not appear anymore. In fact, let us define the functional $J : [1, +\infty) \times L^\infty(X, m) \rightarrow [0, +\infty]$ as

$$J(q, w) := \int_X \frac{|w|^q}{\|w\|_q^q} \log \left(\frac{|w|}{\|w\|_q} \right) \, d\mu.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}
& \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\
& \leq \frac{r'(s)}{r(s)} J(r(s), u(s) - v(s)) \\
& \quad - c(r(s) - 1) \left(\frac{p}{r(s) + p - 2} \right)^{p-1} \frac{\|u(s) - v(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \\
& \quad \times \left[\frac{p^2}{C d \varepsilon} J(r(s) + p - 2, u(s) - v(s)) + \frac{p}{r(s) + p - 2} \frac{\log \varepsilon}{C \varepsilon} \right],
\end{aligned} \tag{3.8}$$

where C is the Sobolev constant appearing in Assumption 1.3 and c is the constant appearing in (3.4).

Step 4. A closed differential inequality for $\log \|u(s) - v(s)\|_{r(s)}$

The difficulty in inequality (3.8) is that it involves L^q norms of the solution for more than one value of q . To overcome such difficulty we proceed as follows. First we choose $\varepsilon = \varepsilon(s)$ depending on u itself so that the coefficients of the entropic terms appearing above are proportional. In fact, let us choose

$$\varepsilon(s) = \frac{cp^2}{Cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s)+p-2} \right)^{p-1} \frac{\|u(s) - v(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}}.$$

The above mentioned inequality then becomes

$$\begin{aligned} & \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\ & \leq \frac{\dot{r}(s)}{r(s)} [J(r(s), u(s) - v(s)) - J(r(s) + p - 2, u(s) - v(s))] \\ & \quad - c(r(s) - 1) \left(\frac{p}{r(s) + p - 2} \right)^p \frac{\|u(s) - v(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \frac{\log \varepsilon(s)}{C\varepsilon(s)}. \end{aligned} \quad (3.9)$$

The first term in the r.h.s. in the above inequality can be estimated by noting that the function $N(q) := \log \|w\|_q^q$ is convex on its domain of definition for every fixed w , and that its derivative equals, for a.e. q , $J(q, w) + \log \|w\|_q$. Such derivative is also monotonically nondecreasing and thus, for any $q_2 \geq q_1$,

$$J(q_1, w) - J(q_2, w) \leq \log \frac{\|w\|_{q_2}}{\|w\|_{q_1}}.$$

This implies, on account of the present choice of ε ,

$$\begin{aligned} & \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\ & \leq \frac{\dot{r}(s)}{r(s)} \log \frac{\|u(s) - v(s)\|_{r(s)+p-2}}{\|u(s) - v(s)\|_{r(s)}} \\ & \quad - \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s) + p - 2} \log \frac{\|u(s) - v(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s) - v(s)\|_{r(s)}^{r(s)}} \\ & \quad - \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s) + p - 2} \\ & \quad \times \log \left[\frac{c_1 p^2}{cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s) + p - 2} \right)^{p-1} \right]. \end{aligned} \quad (3.10)$$

By using Hölder inequality to show that

$$\|u(s) - v(s)\|_{r(s)+p-2} \geq m(X)^{-(p-2)/[r(s)(r(s)+p-2)]} \|u(s) - v(s)\|_{r(s)}$$

and the fact that $p < d$, one can therefore conclude that

$$\begin{aligned} & \frac{d}{ds} \log \|u(s) - v(s)\|_{r(s)} \\ & \leq -\frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{p-2}{r(s)+p-2} \log(\|u(s) - v(s)\|_{r(s)}) \\ & \quad + \frac{\dot{r}(s)}{r(s)} \frac{p-2}{r(s)+p-2} \left(\frac{d}{p} - 1\right) \log m(X) \\ & \quad - \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s)+p-2} \\ & \quad \times \log \left[\frac{c_1 p^2}{cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s)+p-2}\right)^{p-1} \right]. \end{aligned} \quad (3.11)$$

Step 5. Integration of the differential inequality

The above differential inequality (valid for a.e. s) can be integrated if r is chosen explicitly. In fact, we shall set $r(s) = q_0 t / (t - s)$, where $q_0 \geq 2$ is such that the initial datum for the equation at hand belongs to $L^{q_0}(X, m)$ and $t > 0$ is fixed. In fact, let

$$\begin{aligned} p(s) &= \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{p-2}{r(s)+p-2}, \\ q(s) &= -\frac{\dot{r}(s)}{r(s)} \frac{p-2}{r(s)+p-2} \left(\frac{d}{p} - 1\right) \log m(X) \\ & \quad + \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s)+p-2} \\ & \quad \times \log \left[\frac{c_1 p^2}{cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s)+p-2}\right)^{p-1} \right]. \end{aligned} \quad (3.12)$$

Then the function $y(s) := \log \|u(s) - v(s)\|_{r(s)}$ satisfies the following differential inequality:

$$\dot{y}(s) + p(s)y(s) + q(s) \leq 0 \quad \forall s \geq 0,$$

and, therefore, $y(s) \leq \bar{y}(s)$, provided $y(0) \leq \bar{y}(0)$, where

$$\bar{y}(s) = \exp \left[-\int_0^s p(z) dz \right] \left(\bar{y}(0) - \int_0^s q(z) \exp \left[\int_0^z p(z') dz' \right] \right) \quad (3.13)$$

is a solution of the ordinary differential equation

$$\dot{k}(s) + p(s)k(s) + q(s) = 0 \quad \forall s \geq 0.$$

It is elementary to perform the calculations with the present choice of $r(s)$, whose details are left to the reader. The result is

$$\begin{aligned} w(t) &:= \lim_{s \rightarrow t^-} \bar{y}(s) \\ &= \left(\frac{q_0}{q_0 + p - 2} \right)^{d/p} \left[\bar{y}(0) - \frac{1}{p-2} \left(\left(\frac{q_0 + p - 2}{q_0} \right)^{d/p} - 1 \right) \log t \right. \\ &\quad \left. - \frac{d-p}{d} \left[\left(\frac{q_0 + p - 2}{q_0} \right)^{d/p} - 1 \right] \log |N| + K \right], \end{aligned} \quad (3.14)$$

where K depends only upon r_0 , p and d .

Step 6. Hölder continuity

By Step 1 the nonlinear evolution at hand is Markovian, so that, in particular, it is L^q -contractive for any $q \in [2, +\infty]$:

$$\|u(t) - v(t)\|_q \leq \|u(s) - v(s)\|_q,$$

provided $s \leq t$. Therefore for all such s and t

$$\begin{aligned} \|u(t) - v(t)\|_{r(s)} &\leq \|u(s) - v(s)\|_{r(s)} \\ &= \exp[\log \|u(s) - v(s)\|_{r(s)}] = e^{y(s)} \leq e^{\bar{y}(s)}, \end{aligned}$$

whence, letting $s \rightarrow t^-$ and recalling that $r(s) \rightarrow +\infty$ as $s \rightarrow t^-$, we deduce that

$$\begin{aligned} \|u(t) - v(t)\|_\infty &= \lim_{s \rightarrow t^-} \|u(s) - v(s)\|_{r(s)} \leq \lim_{s \rightarrow t^-} e^{\bar{y}(s)} = e^{w(t)} \\ &= C(d, p, q_0) \frac{m(X)^\alpha}{t^\beta} \|u(0) - v(0)\|_{q_0}^\gamma, \end{aligned}$$

where the values of α , β , γ are those appearing in the statement of the theorem.

Step 7. General initial data

We now remove the requirement that the initial data belong to $L^\infty(X, m)$. To this end, given u_0 and v_0 in $L^{q_0}(X, m)$ with $q_0 \geq 2$, take two sequences $\{u_n\}$ and $\{v_n\}$ of elements of $L^\infty(X, m)$, converging in the $L^{q_0}(X, m)$ norm to u_0 and v_0 , respectively. Letting $u_n(t)$ and $v_n(t)$ be the solution to the evolution equation at hand corresponding to the data u_n and v_n , respectively, it follows by the results of the previous step that

$$\|u_n(t) - v_n(t)\|_\infty \leq C(t) \|u_n - v_n\|_{q_0},$$

so that, for all positive t , the sequence $\{u_n(t) - v_n(t)\}$ is bounded in $L^\infty(X, m)$. Possibly by passing to a subsequence, we can assume that such sequence converge, in the weak* topology, to a function $f(t) \in L^\infty(X, m)$ which thus satisfies, by the weak* lower semicontinuity of the L^∞ norm, the bound

$$\|f(t)\|_\infty \leq C(t)\|u_0 - v_0\|_{q_0}.$$

To identify the limit $f(t)$, we notice that the Markov property implies that, if $u(t)$ and $v(t)$ are solutions to the evolution equation considered corresponding to the data u_0 and v_0 , respectively, then

$$\|u_n(t) - u(t)\|_{q_0} \leq \|u_n - u_0\|_{q_0}$$

with a similar inequality holding for $\|v_n(t) - v(t)\|_{q_0}$. Thus $u_n(t) \rightarrow u(t)$ and $v_n(t) \rightarrow v(t)$ in $L^p(X, m)$. Therefore $f(t) = u(t) - v(t)$ and the assertion follows. \square

References

- [1] F. Cipriani, G. Grillo, Uniform bounds for solutions to quasilinear parabolic equations, *J. Differential Equations* 177 (2001) 209–234.
- [2] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [3] N.T. Varopoulos, L. Saloff-Coste, T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, 1992.
- [4] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* 9 (1999) 428–517.
- [5] F. Cipriani, G. Grillo, Nonlinear Dirichlet forms and nonlinear markovian semigroups, Preprint, Dipartimento di Matematica, Politecnico di Milano.
- [6] H. Brezis, *Opérateurs Maximaux Monotones*, North-Holland, Amsterdam, 1973.
- [7] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, Vol. 49, American Mathematical Society, Providence, RI, 1997.
- [8] F. Cipriani, G. Grillo, The Markov property for classes of nonlinear parabolic equations, *Nonlinear Anal.* 47 (2001) 3549–3554.
- [9] P. Bénéilan, C. Picard, Quelques Aspects Nonlinéaires du Principe du Maximum, *Lecture Notes in Mathematics*, Vol. 713, Springer, Berlin, 1979, pp. 1–37.
- [10] E.B. Davies, *Heat Kernel and Spectral Theory*, Cambridge University Press, 1989.
- [11] T. Aubin, Y.Y. Li, On the best Sobolev inequality, *J. Math. Pures Appl.* 78 (1999) 353–387.
- [12] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes in Mathematics, New York, 1999.
- [13] M. Gromov, Carnot–Carathéodory spaces seen from within, in: *Sub-Riemannian Geometry*, Progr. Math., Vol. 144, Birkhäuser, Basel, 1996, pp. 79–323.
- [14] I. Kupka, Géométrie sous-riemannienne, *Sém. Bourbaki* 48 (1995–1996) 817.
- [15] R.S. Strichartz, Sub-Riemannian geometry, *J. Differential Geom.* 24 (1986) 221–263.
- [16] P. Pansu, Une inégalité isopérimétrique sur le groupe de Heisenberg, *C. R. Acad. Sci. Paris* 295 (1982) 127–130.
- [17] L. Capogna, D. Danielli, N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequalities, *Comm. Anal. Geom.* 2 (1994) 203–215.

- [18] B. Franchi, S. Gallot, R. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics, *Math. Ann.* 300 (1994) 557–571.
- [19] J. Heinonen, I. Holopainen, Quasiregular maps on Carnot groups, *J. Geom. Anal.* 7 (1997) 109–148.
- [20] M. Biroli, Schrödinger type and relaxed Dirichlet problem for the subelliptic p -Laplacian, *Potential Anal.* 15 (2001) 1–16.
- [21] N. Garofalo, D.-M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* 49 (1996) 1081–1144.
- [22] D. Jerison, A. Sanchez-Calle, Estimates for the heat kernel for a sum of squares of vector fields, *Indiana Univ. Math. J.* 35 (1986) 835–854.
- [23] D. Danielli, N. Garofalo, D.-M. Nhieu, Trace inequalities for Carnot–Carathéodory spaces and applications, *Ann. Scuola Norm. Sup. Pisa* 27 (1998) 195–252.
- [24] B. Franchi, Ingalits de Sobolev pour des champs de vecteurs lipschitziens, *C. R. Acad. Sci. Paris* 311 (1990) 329–332.
- [25] B. Franchi, C.E. Gutierrez, R.L. Wheeden, Weighted Sobolev–Poincaré inequalities for Grushin type operators, *Comm. Partial Differential Equations* 19 (1994) 523–604.
- [26] G. Lu, Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander’s condition and applications, *Rev. Mat. Iberoamericana* 8 (1992) 367–439.
- [27] G. Lu, The sharp Poincaré inequality for free vector fields: an endpoint result, *Rev. Mat. Iberoamericana* 10 (1994) 453–466.
- [28] P. Maheaux, L. Saloff-Coste, Analyse sur le boules d’un opérateurs sous-elliptique, *Math. Ann.* 303 (1995) 713–740.
- [29] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Math. Res. Notices* (1992) 27–38.
- [30] P. Hajlasz, P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* 145 (2000) 688.
- [31] B. Franchi, G. Lu, R. Wheeden, A relationship between Poincaré type inequality and representation formulas in spaces of homogeneous type, *Internat. Math. Res. Notices* (1996) 1–14.
- [32] B. Franchi, R. Wheeden, Some remarks about Poincaré type inequality and representation formulas in spaces of homogeneous type, *J. Ineq. Appl.* 3 (1999) 1–14.
- [33] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.
- [34] F. Cipriani, J.L. Sauvageot, Derivations as square root of Dirichlet forms, Preprint, Dipartimento di Matematica, Politecnico di Milano.
- [35] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Markov Processes*, De Gruyter, Berlin, 1994.
- [36] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.* 97 (1976) 1061–1083.